# Analysis of Halbach Segmented Pure Permanent Magnet Quadrupole

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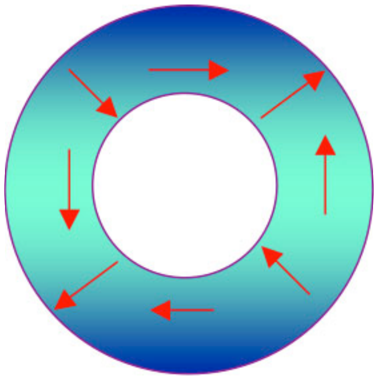
# The ideal Halbach quad

- Imagine a permanent magnet geometry in which the magnetization varies continuously with $\phi$, as shown

$$\vec{M} = M_0\left(\hat{\rho}\sin(2\phi) + \hat{\phi}\cos(2\phi)\right)$$



- This model problem can be used to generate the segmented-piece solutions
- In order to analyze this geometry, it is best to introduce Amperian currents…

# Amperian currents:

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- The equivalent currents are found in a sheet on the inner and outer radius ($r_1$ and $r_2$)

$$\vec{K} = \vec{M} \times \hat{n}$$

$$K_z = \pm M_0 \cos(2\phi), \ \ \rho = r, R$$

- Inside of the material, there is a bulk equivalent current density

$$\vec{J} = \vec{\nabla} \times \vec{M}$$

$$J_z = \frac{3M_0}{\rho} \cos(2\phi)$$

# Creating a "Green function"

- To solve for the fields due to these current arrays we first note that for a current sheet at $\rho = a$ with $\cos(2\phi)$ dependence, the solutions for the vector potential inside and outside of the sheet are

$$A_z = A_2 \cos(2\phi) \begin{cases} \rho^2, & \rho < a \\ \dfrac{a^4}{\rho^2}, & \rho > a \end{cases}$$

- The magnetic fields of this pure quadrupole are given by

$$\vec{B} = -2A_2 \begin{cases} \rho \sin(2\phi)\hat{\rho} + \rho \cos(2\phi)\hat{\phi}, & \rho < a \\ \dfrac{a^4}{\rho^3}\sin(2\phi)\hat{\rho} - \dfrac{a^4}{\rho^3}\cos(2\phi)\hat{\phi}, & \rho > a. \end{cases}$$

# Connecting field strength to the current sheet

- The discontinuity in the azimuthal field is created by the current sheet at $\rho=a$; $\Delta B_\phi(\phi) = \mu_0 K_z(\phi) = \mu_0 K_0 \cos(2\phi)$

$$= 4 A_2 a \cos(2\phi)$$

- Thus, $A_2 = \frac{\mu_0 K_0}{4a}$ or in terms of the field gradient,

$$B' = 2A_2 = \frac{\mu_0 K_0}{2a} \propto \frac{K_0}{\rho}$$

for the single current sheet.

- For a radially distributed current density $J_z = J_0(\rho)\cos(2\phi)$ (between $r_1$ and $r_2$), this result can be generalized to give

$$B' = \frac{\mu_0}{2} \int_{r_1}^{r_2} J_0(\rho)\, d\rho$$

# Application to permanent magnet case

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- The contributions due to the boundaries at $\rho = r_1, r_2$:

$$B' = \frac{\mu_0 M_0}{2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = \frac{B_r}{2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

where $B_r$ is the peak remanent magnetic field of the PM material.

- The contributions to the field gradient due to the bulk of the magnetic material are

$$B' = \frac{3\mu_0 M_0}{2}\int_{r_1}^{r_2}\frac{d\rho}{\rho^2} = \frac{3}{2}B_r\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

- The total field gradient for the pure quadrupole is

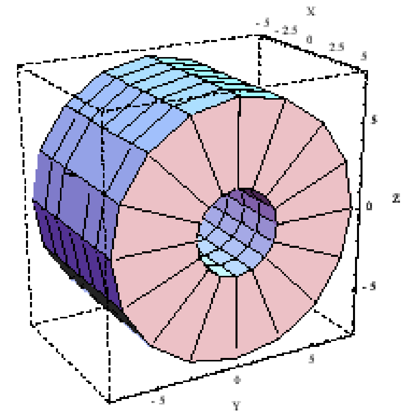
$$B' = 2B_r\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

For our case, a PMQ with 5 mm ID and $B_r$=1.2 T, we have

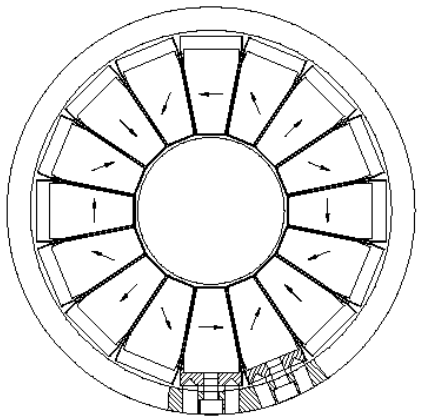
$$B' = 600\ \mathrm{T/m!}$$

# Segmented magnets

- The pure quadrupole is impossible to build, but may be well-approximated using uniformly magnetized pieces
- Segmented slices (pizza-pie with bites taken out) can be analyzed using our methods
- Fourier analysis gives equivalent Amperian current densities at desired multipole, *and* harmonics!



**UCLA PMQ RADIA model**



**CESR collider final focus PMQ cross-section**

# Fourier analysis of segmented PMQ

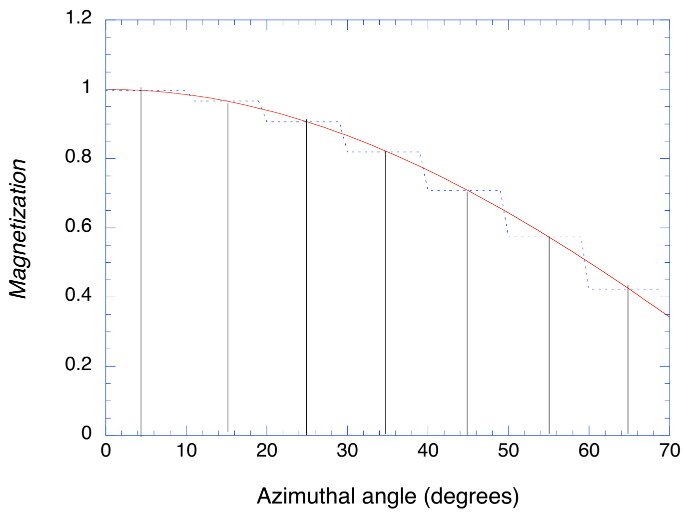
- $M$ segments used (e.g. 16 for our case)
- Interior region (straightforward, summation of $\delta$-functions)

$$J_z = \frac{3}{\rho} \sum_{n=2,4,6...}^{\infty} a_n \cos(n\phi)$$

$$a_n = \frac{1}{2\pi} \sum_{m=1}^{M} \cos\left(\frac{2\pi}{M}\left[m - \tfrac{1}{2}\right]\right) \cos\left(\frac{2\pi n}{M}\left[m - \tfrac{1}{2}\right]\right)$$

$$a_n = \cos^n\left(\frac{\pi}{M}\right) \sin\left(\frac{n\pi}{M}\right) \frac{M}{n\pi}, \quad \text{where } n = 2 + jM, \ j = 0,1,2...$$

- The inner/outer radii surfaces produce analogous results




This procedure approximates the sinusoidal magnetization with a series of δ-functions

# PMQ strength and higher terms

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- Fundamental strength is derated from perfect quadrupole by Fourier harmonics

$$a_2 = \cos^2\left(\frac{\pi}{M}\right)\sin\left(\frac{2\pi}{M}\right)\frac{M}{2\pi}$$

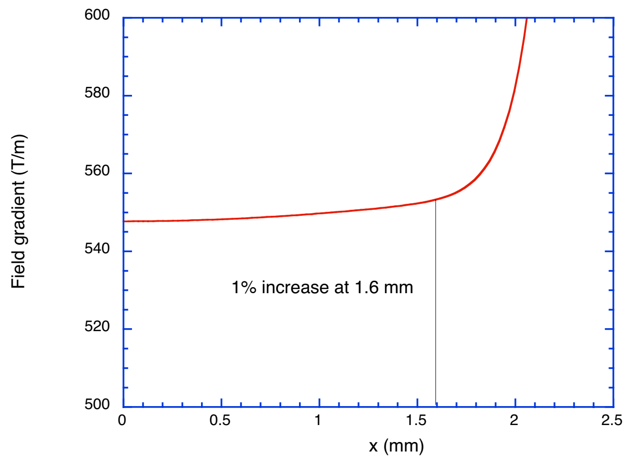
| $M$ (segments) | $a_2$ |
|---|---|
| 8 | 0.77 |
| 12 | 0.89 |
| 16 | 0.94 |

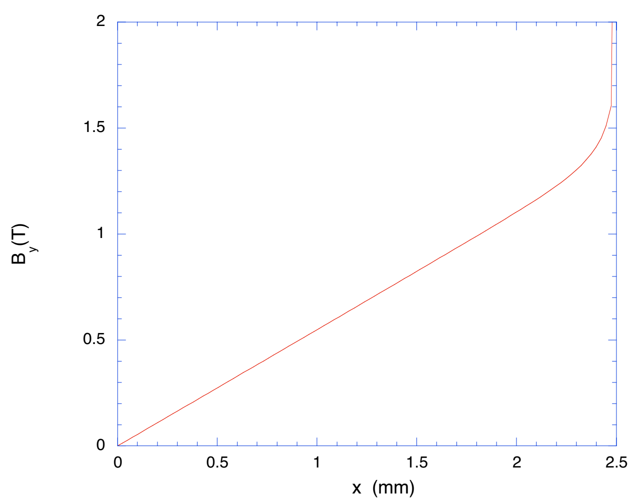
- The next higher term is $n=18$! Relative strength is

$$\frac{a_n}{a_2} = \cos^{n-2}\left(\frac{\pi}{M}\right)\left(\frac{1-\left(r_1/r_2\right)^{n-1}}{1-\left(r_1/r_2\right)^{2}}\right)$$

- Term is not large, and order is very high…

# Field gradient model




- Fixed RADIA solution gives 550 T/m with $B_r$=1.2 T
- Very good linearity
- Model prediction:

  600 T/m
- Deviations due to:
  - finite longitudinal (3D) effects
  - demagnetization

# Measurements

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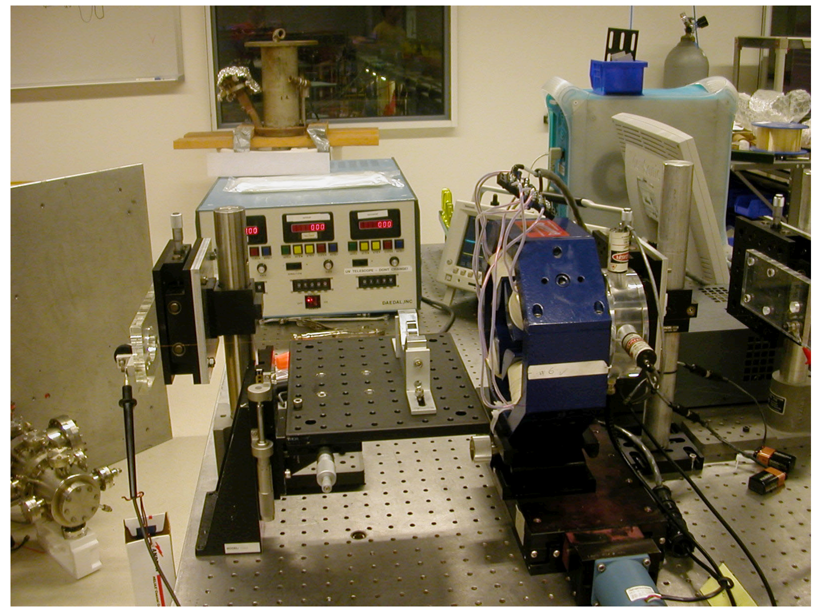
- Hall probe scan: 550 T/m
- Consistency with beam focus tuning (TRACE3D): 550 T/m
- Pulsed wire: 475 T/m
  - Is pulse short enough for first integral? Needs to be a $\delta$-function
  - Frequency dependence of wire



Calibration setup for
PMQ gradient measurement