Pb 1a) - Show that after acceleration in a gun (so that β = 1 and $p_z \sim \gamma$) the minimal longitudinal RF emittance is given by:

$$\varepsilon_z^{rf} = \frac{1}{2k}\left(\gamma_f - 1\right)\sqrt{\left\langle(\Delta\varphi)^4\right\rangle\left\langle(\Delta\varphi)^2\right\rangle}$$

where k=2π/λ and γ_f is the value of <γ> at the gun exit.

1b) - Show that for a Gaussian distribution the minimal longitudinal RF emittance is given by:

$$\varepsilon_z^{rf} = \frac{1}{2}\left(\gamma_f - 1\right)k^2\sigma_z^3$$

1a) The definition of longitudinal emittance is:

$$\varepsilon_z = \frac{1}{k}\sqrt{\left\langle(\Delta\gamma)^2\right\rangle\left\langle(\Delta\varphi)^2\right\rangle - \left\langle\Delta\gamma\Delta\varphi\right\rangle^2}$$

Setting <ϕ>=π/2 to minimize the transverse RF emittance we have up to II order:

$$\Delta\gamma = \gamma - \langle\gamma\rangle = -\alpha\Delta\varphi - \frac{1}{2}\left(\gamma_f - 1\right)(\Delta\varphi)^2 + \ldots.$$

$$\left\langle(\Delta\gamma)^2\right\rangle = -\alpha^2\left\langle(\Delta\varphi)^2\right\rangle + \frac{1}{4}\left(\gamma_f - 1\right)^2\left\langle(\Delta\varphi)^4\right\rangle + \alpha\left(\gamma_f - 1\right)\left\langle(\Delta\varphi)^3\right\rangle$$

$$\Delta\gamma\Delta\varphi = -\alpha(\Delta\varphi)^2 - \frac{1}{2}\left(\gamma_f - 1\right)(\Delta\varphi)^3$$

$$\left\langle\Delta\gamma\Delta\varphi\right\rangle^2 = -\alpha^2\left\langle(\Delta\varphi)^2\right\rangle^2 + \frac{1}{4}\left(\gamma_f - 1\right)^2\left\langle(\Delta\varphi)^3\right\rangle^2 + \alpha\left(\gamma_f - 1\right)\left\langle(\Delta\varphi)^2\right\rangle\left\langle(\Delta\varphi)^3\right\rangle$$

hence:

$$\varepsilon_z^{rf} = \frac{1}{2k}\left(\gamma_f - 1\right)\sqrt{\left\langle(\Delta\varphi)^4\right\rangle\left\langle(\Delta\varphi)^2\right\rangle - \left\langle(\Delta\varphi)^3\right\rangle^2}$$

for symmetric distributions we have

$$\left\langle(\Delta\varphi)\right\rangle = \left\langle(\Delta\varphi)^3\right\rangle = 0$$

thus:

$$\varepsilon_z^{rf} = \frac{1}{2k}\left(\gamma_f - 1\right)\sqrt{\left\langle(\Delta\varphi)^4\right\rangle\left\langle(\Delta\varphi)^2\right\rangle}$$

1b) For a Gaussian distribution $\left\langle(\Delta\varphi)^2\right\rangle = k^2\sigma_z^2$ and $\left\langle(\Delta\varphi)^4\right\rangle = k^4\sigma_z^4$ so that by substituting in the previous equation we have $\varepsilon_z^{rf} = \frac{1}{2}\left(\gamma_f - 1\right)k^2\sigma_z^3$

Pb 2 - Show that for a uniform distribution in a cylinder of radius a and length L, the relevant moments for the distribution are:

$$\left\langle x^2\right\rangle = \frac{a^2}{4}, \qquad \left\langle(\Delta\varphi)^2\right\rangle = \frac{k^2L^2}{12}, \qquad \left\langle(\Delta\varphi)^4\right\rangle = \frac{k^4L^4}{80}$$

and the transverse RF emittance is:

$$\varepsilon_x^{rf} = \frac{\alpha a^2 k^3 L^2}{4\sqrt{6!}}$$

$$\left\langle x^2\right\rangle = \frac{1}{\pi a^2 L}\int_{-L/2}^{L/2}\int_{0}^{2\pi}\int_{0}^{a}\left(r\cos\vartheta\right)^2 r\,dr\,d\vartheta\,dz = \frac{a^2}{4}$$

$$\left\langle(\Delta\varphi)^2\right\rangle = \frac{1}{\pi a^2 L}\int_{-L/2}^{L/2}\int_{0}^{2\pi}\int_{0}^{a}\left(kz\right)^2 r\,dr\,d\vartheta\,dz = \frac{k^2L^2}{12}$$

$$\left\langle(\Delta\varphi)^4\right\rangle = \frac{1}{\pi a^2 L}\int_{-L/2}^{L/2}\int_{0}^{2\pi}\int_{0}^{a}\left(kz\right)^4 r\,dr\,d\vartheta\,dz = \frac{k^4L^4}{80}$$

Now we can compute the transverse RF emittance, setting $\langle\phi\rangle=\pi/2$ and substituting the computed moments in:

$$\varepsilon_x^{rf} = \alpha\kappa\,\frac{\left\langle x^2\right\rangle}{2}\sqrt{\left\langle(\Delta\varphi)^4\right\rangle - \left\langle(\Delta\varphi)^2\right\rangle^2} = \frac{\alpha a^2 k^3 L^2}{4\sqrt{6!}}$$