Overview of Beam Physics

Lecture 2

Electrodynamics

- Start with Maxwell equations (mks)

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \cdot \vec{D} = \rho_e \qquad \vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}_e \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Where the fields obey the constitutive relations

$$\vec{D} = \varepsilon(\vec{D})\vec{E} \qquad \vec{B} = \mu\left(\vec{H}\right)\vec{H}$$

- Continuity of sources implied $\vec{\nabla} \cdot \vec{J}_e + \frac{\partial \rho_e}{\partial t} = 0$
- Charged particles obey the Lorentz force equation,

$$\frac{d\vec{p}}{dt} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

With a generalization of the momentum *p*.

Relativistic dynamical quantities

- Velocity normalized to speed of light $\vec{v} \equiv \vec{\beta} c$
- Momentum is defined relativistically as

$$\vec{p} = \gamma m_0 \vec{v} \equiv \vec{\beta} \gamma m_0 c$$

with m_0 being the rest mass (nonrel. limit), and

$$\gamma \equiv \left(1 - \left(\vec{v}^{\,2} / c^2\right)\right)^{-1/2}$$

- The energy U and momentum are related by 4-vector invariant length (like space-time)

$$\vec{p}^{\,2} c^2 - U^2 = -\left(m_0 c^2\right)^2 \qquad U = \gamma m_0 c^2$$

Relativistic equations of motion

- Purely magnetic forces yield no change in energy, as force is normal to instantaneous motion, $dU = \vec{v} \cdot d\vec{p}$
- Transverse equation of motion has larger inertial mass

$$\gamma m_0 \frac{d\vec{v}}{dt} = q\left(\vec{v} \times \vec{B}\right)$$

- Longitudinal acceleration with electric field gives change in γ, different scaling of inertial mass $\gamma^3 m_0 \frac{dv_{\parallel}}{dt} = qE_0$

EM Fields in special relativity

- Longitudinal fields are "Lorentz invariant"
- Transverse fields (*i.e.* self fields in beam)

$$\vec{E}_\perp = \gamma\left(\vec{E}'_\perp - \vec{v} \times \vec{B}'_\perp\right)$$

$$\vec{B}_\perp = \gamma\left(\vec{B}'_\perp + \tfrac{1}{c^2}\vec{v} \times \vec{E}'_\perp\right)$$

- In beam rest frame $\vec{B}'_\perp = 0$ and

$$\vec{F}_\perp = q\left(\vec{E}_\perp + \vec{v} \times \vec{B}_\perp\right) = q\vec{E}_\perp\left(1 - \beta^2\right) = \frac{q\vec{E}_\perp}{\gamma^2}$$

"extra" factors of inertial mass…

Hamiltonians in special relativity

- A Hamiltonian is a function of coordinates and "canonical momentum" which gives the equations of motion $\dot{x}_i = \frac{\partial H}{\partial p_i}, \;\; \dot{p}_i = -\frac{\partial H}{\partial x_i}.$
- It is constructed from a *Lagrangian*

$$H(\vec{x}, \vec{p}) = \vec{p} \cdot \dot{\vec{x}} - L \qquad p_i \equiv \frac{\partial L}{\partial \dot{x}_i}$$

- A Hamiltonian can rigorously give equations of motion, and *constants* of the motion, *i.e. H,*

$$\dot{H} = \frac{\partial H}{\partial t};\; H \text{ independent of time is constant.}$$

Relativistic dynamics

- Need potentials $\left(\phi_{\mathrm{e}},\vec{A}\right)$ for variational analysis

$$\vec{E} = -\vec{\nabla}\phi_{\mathrm{e}} - \frac{\partial \vec{A}}{\partial t} \qquad \vec{B} = \vec{\nabla} \times \vec{A}$$

- The Lorentz force can be written as

$$\vec{F}_L = \frac{d\vec{p}}{dt} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

$$= q\left[-\vec{\nabla}\phi_{\mathrm{e}} - \frac{\partial \vec{A}}{\partial t} - \left(\vec{v} \cdot \vec{\nabla}\right)\vec{A}\right] = q\left[-\vec{\nabla}\phi_{\mathrm{e}} - \frac{d\vec{A}}{dt}\right]$$

- Relations between energy/momenta and potentials can be seen

$$p_{c,i} = p_i + qA_i \qquad H = U + q\phi_e$$

The relativistic Hamiltonian

- From the 4-vector relation of energy and momentum,

$$\left(H - q\phi_{\mathrm{e}}\right)^2 = \left(\vec{p}_c - q\vec{A}\right)^2 c^2 + \left(m_0 c^2\right)^2$$

or

$$H = \sqrt{\left(\vec{p}_c - q\vec{A}\right)^2 c^2 + \left(m_0 c^2\right)^2} + q\phi_{\mathrm{e}}$$

- Canonical variables simply related to familiar *mechanical* variables (U,p)

Transformations of Hamiltonian

- Transform to new canonical variables to make the Hamiltonian constant?
- Example: wave with phase velocity v_φ
- Galilean transformation $\zeta = z - v_\varphi t \quad p_\zeta = p_z$
- New Hamiltonian $\tilde{H}(\zeta, p_\zeta) = H(\zeta, p_\zeta) - v_\phi p_\zeta .$
- Example: traveling wave in accelerator…

Traveling wave Hamiltonian

- Field and vector potential

$$E_z\left(z - v_\phi t\right) = -\frac{\partial A_z}{\partial t} = E_0 \sin\left[k_z\left(z - v_\phi t\right)\right], \qquad A_z\left(z - v_\phi t\right) = -\frac{E_0}{k_z v_\phi}\cos\left[k_z\left(z - v_\phi t\right)\right],$$

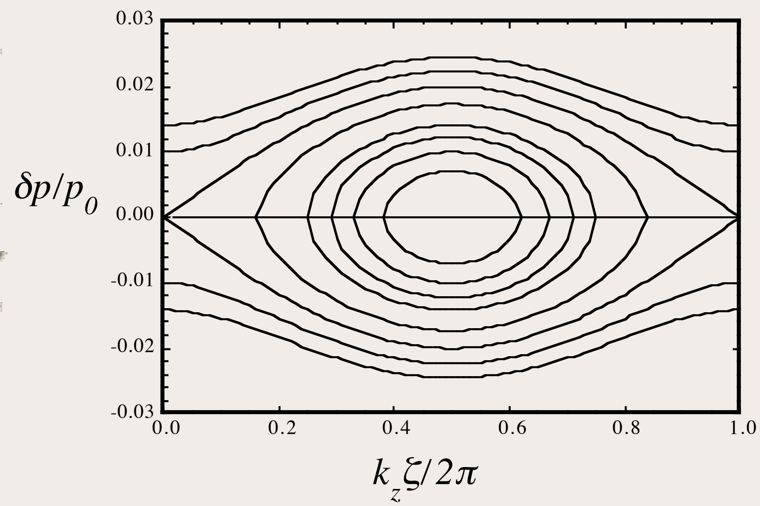
- 1D COM Hamiltonian:

$$H = \sqrt{\left(p_{z,c} + \frac{qE_0}{k_z v_\phi}\cos\left[k_z\left(z - v_\phi t\right)\right]\right)^2 c^2 + \left(m_0 c^2\right)^2}$$

- In terms of mechanical momentum for plotting ("algebraic" Hamiltonian, *not* for equations of motion)

$$\tilde{H}\left(\zeta, p_\zeta\right) = \sqrt{{p_\zeta}^2 c^2 + \left(m_0 c^2\right)^2} - v_\varphi\, p_\zeta + \frac{qE_0}{k_z}\cos\left[k_z \zeta\right]$$

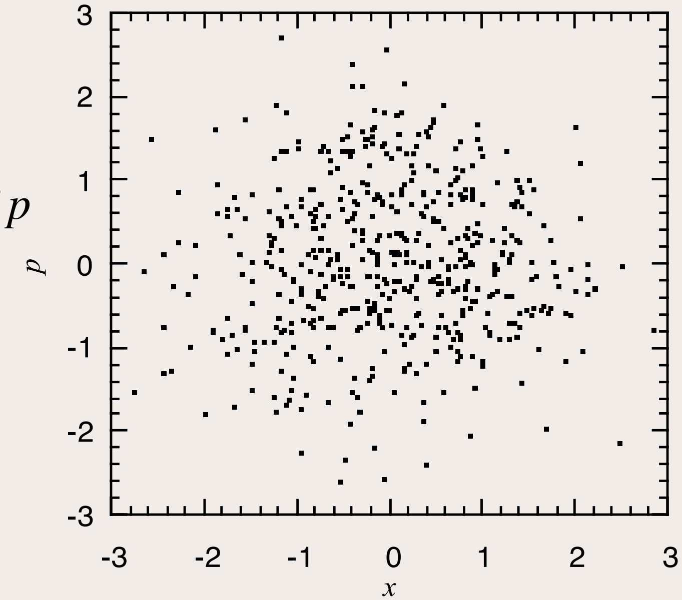
Phase plane plots



- With algebraic form, for a given *H,* choose ζ, can determine *p*.
- Plot motion along constant *H* contours
- Example: longitudinal motion in small "potential" case, typical of ion linacs and circular accelerators

Phase space, general considerations

- Phase space density distribution: $f(\vec{x},\vec{p},t)$
- Number of particles near a phase space point is $f(\vec{x},\vec{p},t)d^3xd^3p$
- In beam physics, we often will have 2D projections of phase space, *i.e.* (x,p_x)
- Motion uncoupled in *x,y,* and *z*.



Particle distribution in phase space projection, approximated by *f*

Liouville theorem

- Vlasov equation $\frac{df}{dt} = \frac{\partial f}{\partial t} + \dot{\vec{x}} \cdot \vec{\nabla}_{\vec{x}} f + \dot{\vec{p}} \cdot \vec{\nabla}_{\vec{p}} f$
- For Hamiltonian systems:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_i \left(\frac{dx_i}{dt}\frac{\partial f}{\partial x_i} + \frac{dp_i}{dt}\frac{\partial f}{\partial p_i} \right)$$
$$= \frac{\partial f}{\partial t} + \sum_i \left(\frac{\partial H}{\partial p_i}\frac{\partial f}{\partial x_i} - \frac{\partial H}{\partial x_i}\frac{\partial f}{\partial p_i} \right)$$
$$= \frac{\partial f}{\partial t} + \sum_i \left(\frac{\partial H}{\partial p_i}\frac{\partial H}{\partial x_i}\frac{df}{dH} - \frac{\partial H}{\partial x_i}\frac{\partial H}{\partial p_i}\frac{df}{dH} \right) = \frac{\partial f}{\partial t}$$

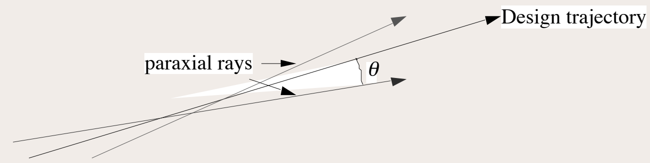
- If particle number is conserved $\frac{\partial f}{\partial t} = 0$
- Conservation of phase space density: $\frac{df}{dt} = 0.$

Design trajectory and paraxial rays

- Trajectories measured with respect to idealized design
- Small angles assumed; paraxial rays

$$p_{x,y} << p_z \cong |\vec{p}|$$

- Equations of motion parameterized using *z* as independent variable instead of *t:* $x' = \frac{dx}{dz} = \frac{1}{v_z}\frac{dx}{dt} = \frac{\dot{x}}{v_z}$



Example: equations of motion in magnetic quadrupole

- Magnetic field (normal)

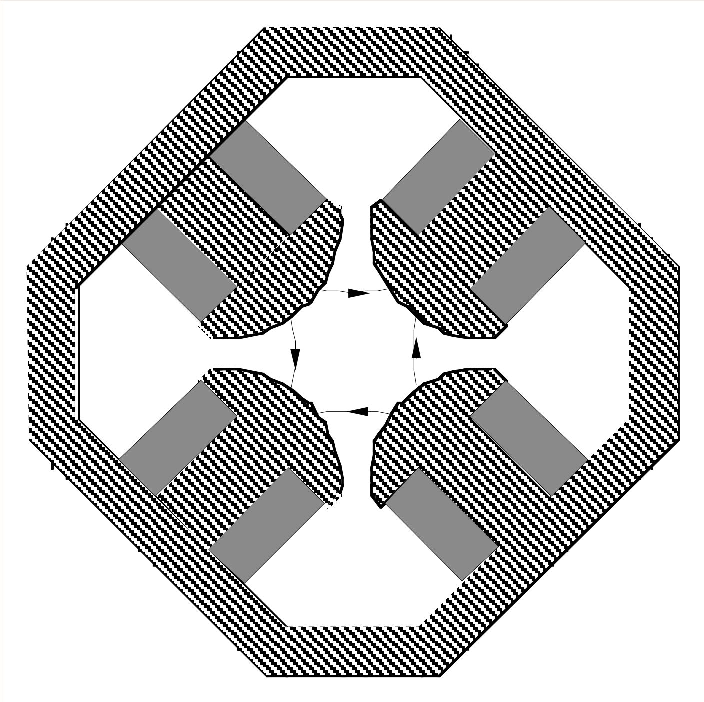
$$\vec{B}_2 = B'(y\hat{x} + x\hat{y})$$

- Magnetic force

$$\vec{F}_\perp = qv_z\vec{z} \times \vec{B}_2 = qv_z B'(y\hat{y} - x\hat{x})$$

- Equation of motion

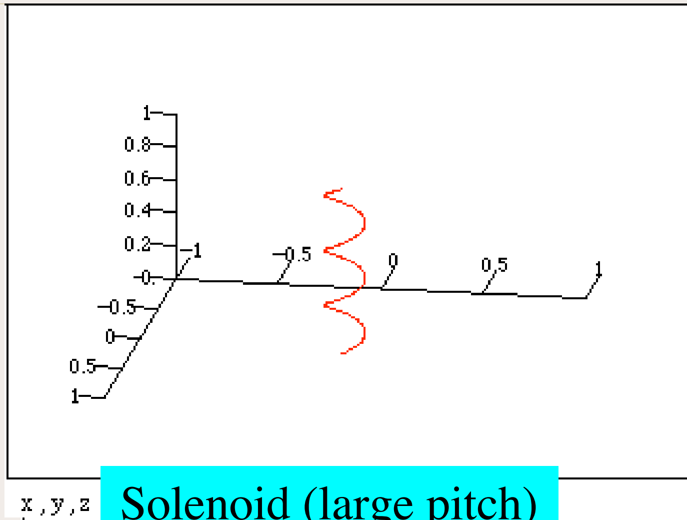
$$x'' = \frac{F_x}{\gamma m_0 v_0^2} = -\frac{qB'}{p_0}x \qquad y'' = \frac{F_y}{\gamma m_0 v_0^2} = \frac{qB'}{p_0}y$$



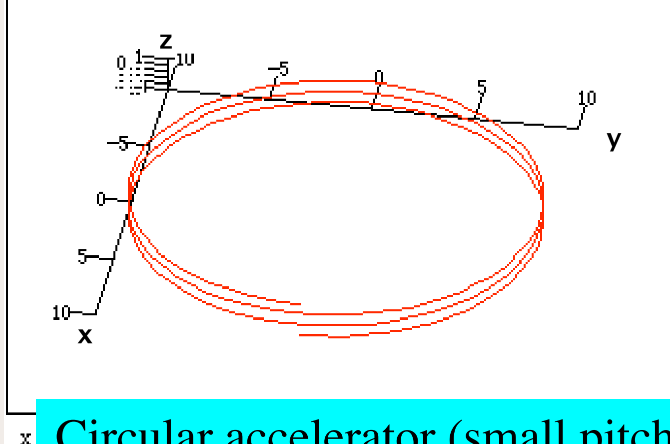
- Simple harmonic oscillator form: $x'' + \kappa_0^2 x = 0 \quad \kappa_0^2 \equiv qB'/p_0 = B'/BR$

Solution: $x = x_0 \cos[\kappa_0(z - z_0)] + \frac{x_0'}{\kappa_0}\sin[\kappa_0(z - z_0)]$

Motion in uniform magnetic field



Solenoid (large pitch)



Circular accelerator (small pitch)

- Field $\vec{B} = B_0\hat{z}$
- Lorentz force

$$\frac{dp_z}{dt} = 0,$$

$$\frac{d\vec{p}_\perp}{dt} = q\left(\vec{v}_\perp \times \vec{B}\right) = \frac{qB_0}{\gamma m_0}(\vec{p}_\perp \times \hat{z}).$$

- Uniform motion in z
- Transverse circle at cyclotron frequency.

$$\frac{d^2 v_x}{dt^2} + \omega_c^2 v_x = 0, \quad \frac{d^2 v_y}{dt^2} + \omega_c^2 v_y = 0. \quad \omega_c \equiv \frac{qB_0}{\gamma m_0}$$

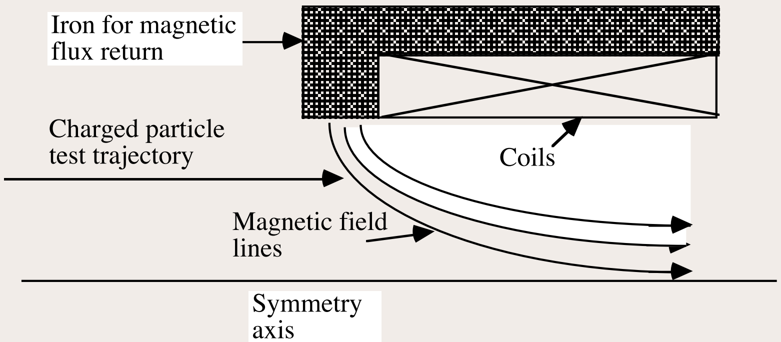
- Radius related to $p_\perp$

$$p_\perp(\text{MeV/c}) = 299.8 \cdot B_0(\text{T})R(\text{m})$$

Define: $BR \equiv p_\perp(\text{MeV/c})/299.8$

- Helix in general, angle $p_z / p_\perp$

Solenoid focusing



- Particles pick up $p_\perp$ in fringe field

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho B_\rho = -\frac{\partial B_z}{\partial z} \quad \text{or} \quad B_\rho \cong -\frac{\rho}{2}\frac{\partial B_z}{\partial z}\bigg|_{\rho=0}$$

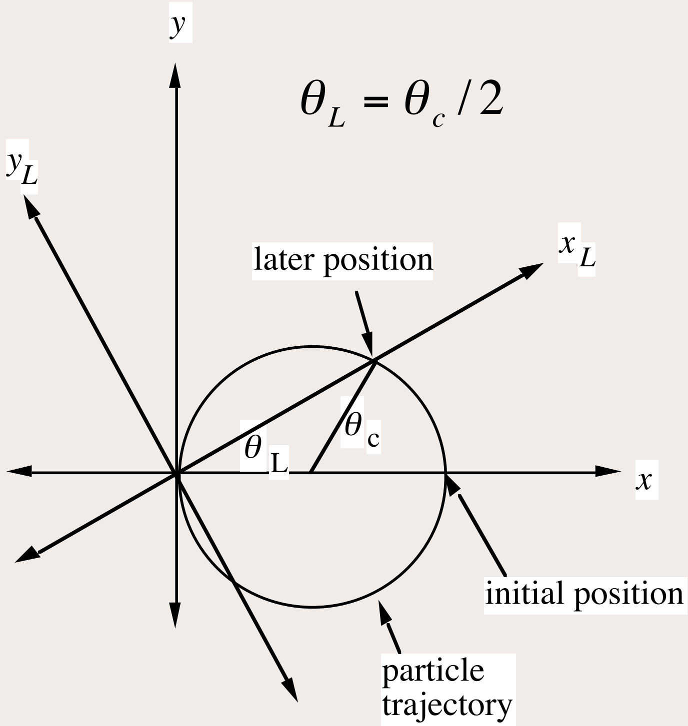
- Integrate, to find

$$\Delta p_\phi \cong q\int_{t_1}^{t_2} v_z B_\rho dt = q\int_{z_1}^{z_2} B_\rho dz = -q\frac{\rho_0}{2}\int_{z_1}^{z_2}\frac{\partial B_z}{\partial z}\bigg|_{\rho=0} dz = -q\frac{\rho_0}{2}\int_{z_1}^{z_2}\frac{dB_z}{dz}\bigg|_{\rho=0} dz$$

$$= -q\frac{\rho_0}{2}\left[B_z(z_2) - B_z(z_1)\right] = -q\frac{\rho_0}{2}B_0$$

- Result known as Busch’s theorem
- Assume particles are “born” in region with no magnetic field…

The Larmor Frame



- Note radius of curvature *R* is 1/2 of offset *ρ*. Particle passes through origin!
- A frame defined by particle and origin rotates with *Larmor frequency*

$$\omega_L \equiv \frac{d\theta_L}{dt} = \frac{\omega_c}{2} = \frac{qB_0}{2\gamma m_0}$$

- In Larmor frame, equations of motion are simple harmonic

$$\ddot{x}_L + \omega_L^2 x_L = 0, \quad \ddot{y}_L + \omega_L^2 y_L = 0, \quad \Longrightarrow \quad x_L'' + k_L^2 x_L = 0, \quad y_L'' + k_L^2 y_L = 0, \quad k_L = \frac{\omega_L}{v_z} \cong \frac{qB_0}{2p} = \frac{B_0}{2BR}$$

Distributions and equilibrium

- A distribution can be in equilibrium under application of *linear* force
- Separability: $f(\vec{x},\vec{p}) = Nf_x(x,p_x)f_y(y,p_y)f_z(z,p_z)$ $\longrightarrow$ $f_x(x,p_x) = X(x)P(p_x)$
- Vlasov equilibrium $\dot{x}\frac{dX}{dx}P + F_x X\frac{dP}{dp_x} = 0$
- Separated equations: $\frac{1}{XF_x(x)}\frac{dX}{dx} = -\frac{\gamma_0 m_0}{pP}\frac{dP}{dp_x} = \lambda_s$
- Momentum solution $P(p_x) = C_p \exp\left(-\frac{p_x^2}{2\gamma_0 m_0 k_B T_x}\right) \equiv C_p \exp\left(-\frac{p_x^2}{2\sigma_{p_x}^2}\right)$
- Solution for linear restoring force in *x*:

$$X(x) = C_x \exp\left(-\frac{\gamma_0 m_0 v_0^2 \kappa_0^2 x^2}{2k_B T_x}\right) \equiv C_x \exp\left(-\frac{x^2}{2\sigma_x^2}\right)$$

- Self-forces (space-charge) are nonlinear in *x*, *Maxwell-Vlasov* equilibria not Gaussian.

RMS envelope equation

- Want information on extent of distributions
- Look at RMS envelope

$$\sigma_x^2 = \left\langle x^2 \right\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 f_x(x,x')\,dx\,dx',$$

- Take derivatives

$$\frac{d\sigma_x}{dz} = \frac{d}{dz}\sqrt{\left\langle x^2 \right\rangle} = \frac{1}{2\sigma_x}\frac{d}{dz}\left\langle x^2 \right\rangle$$

$$= \frac{1}{2\sigma_x}\frac{d}{dz}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 f_x(x,x')\,dx\,dx'$$

$$= \frac{1}{\sigma_x}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xx' f_x(x,x')\,dx\,dx' = \frac{\sigma_{xx'}}{\sigma_x}.$$

$$\frac{d^2\sigma_x}{dz^2} = \frac{d}{dz}\frac{\sigma_{xx'}}{\sigma_x} = \frac{1}{\sigma_x}\frac{d\sigma_{xx'}}{dz} - \frac{\sigma_{xx'}^2}{\sigma_x^3}$$

$$= \frac{1}{\sigma_x}\frac{d}{dz}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xx' f_x(x,x')\,dx\,dx' - \frac{\sigma_{xx'}^2}{\sigma_x^3}$$

$$= \frac{\sigma_{x'}^2 + \left\langle xx'' \right\rangle}{\sigma_x} - \frac{\sigma_{xx'}^2}{\sigma_x^3}$$

RMS Emittance and the Envelope

- The RMS envelope equation can be put in standard form by noting that the *rms emittance*

$$\varepsilon_{x,\mathrm{rms}}^2 = \langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2$$

is constant under of linear forces $x'' + \kappa_x^2 x = 0$

- Also, with this assumption, $\langle xx'' \rangle = -\kappa_x^2 \langle x^2 \rangle = -\kappa_x^2 \sigma_x^2$

and

$$\sigma_x'' + \kappa_x^2 \sigma_x = \frac{\varepsilon_{x,\mathrm{rms}}^2}{\sigma_x^3}.$$

- The emittance enters the envelope equation as a pressure-like term
- Example solution: equilibrium $\sigma_{\mathrm{eq}} = \sqrt{\frac{\varepsilon_{x,rms}}{\kappa_x}}$

Inclusion of Acceleration

- In electron sources we have strong acceleration
- Acceleration introduces "adiabatic" damping

$$\frac{d^2x}{dz^2} = \frac{d}{dz}\frac{p_x}{p_z} = -\kappa_x^2 x + \frac{(\beta\gamma)}{\beta\gamma}x' \ , \quad (\beta\gamma)' = \frac{F_z}{p}$$

- This enters into the envelope as

$$\frac{d^2\sigma_x}{dz^2} = \frac{d}{dz}\frac{\sigma_{xx'}}{\sigma_x} = \frac{1}{\sigma_x}\frac{d\sigma_{xx'}}{dz} - \frac{\sigma_{xx'}^2}{\sigma_x^3}$$

$$= \frac{1}{\sigma_x}\left[\sigma_{x'}^2 - \kappa_x^2\sigma_x^2 - \frac{(\beta_0\gamma_0)'}{\beta_0\gamma_0}\sigma_{xx'}\right] - \frac{\sigma_{xx'}^2}{\sigma_x^3}$$

- In standard form $$\frac{d^2\sigma_x}{dz^2} + \frac{(\beta\gamma)'}{\beta\gamma}\frac{d\sigma_x}{dz} + \kappa_x^2\sigma_x = \frac{\varepsilon_{n,x}^2}{(\beta\gamma)^2\sigma_x^3}$$

  where we have introduced the *normalized emittance*

$$\varepsilon_{n,x} \equiv \beta\gamma\varepsilon_{x,rms}$$

Busch's theorem: magnetization emittance

- If particles are "born" in magnetic field, then they have canonical angular momentum. Upon leaving field, they have rms transverse momentum $\sigma_{p\perp} \cong \frac{qB_0}{2}\sigma_{x(y)}$
- This can be translated to normalized emittance

$$\varepsilon_{n.x} \approx \frac{\sigma_{p\perp}}{m_0 c}\sigma_x \cong \frac{qB_0}{2m_0 c}\sigma_x^2 \approx 150 B_0(T)\sigma_x^2$$

Reading references

- Review Chapter 1 sections (all) concerning dynamics and phase space
- Review Chapter 2 sections (1,3-6) concerning motion in magnetic fields and linear focusing
- Review Chapter 5 sections (1-3, 5) concerning distributions and envelopes