1. The derivation is as follows:

$$\frac{\partial}{\partial t} f(\vec{x},\vec{p}) = \frac{\partial}{\partial t} g\left[H(\vec{x},\vec{p})\right]$$
$$= g'\left[\frac{dH}{dt}\right] = g'\left[\frac{\partial H}{\partial p}\dot{p} + \frac{\partial H}{\partial x}\dot{x}\right]$$
$$= g'\left[\frac{\partial H}{\partial p}\frac{\partial H}{\partial x} - \frac{\partial H}{\partial x}\frac{\partial H}{\partial p}\right] = 0.$$

This also explicitly show that $\frac{dH}{dt}$=0 for a time-independent Hamiltonian. The form of *transverse* (only x dimension) Hamiltonian was used in the example in class is simply $g\left[H(x,p_x)\right] = \exp - \frac{H}{k_B T_x}$. This H can be deduced by looking at the forms of the equilibria,

$$f_x(x,(p_x)) = X(x)P(p_x) = C_x C_p \exp\left(-\frac{\gamma_0 m_0 v_0^2 \kappa_0^2 x^2}{2k_B T_x} - \frac{p_x^2}{2\gamma_0 m_0 k_B T_x}\right),$$
$$= C_x C_p \exp\left(-\frac{1}{k_B T_x}\left[\frac{\gamma_0 m_0 v_0^2 \kappa_0^2 x^2}{2} + \frac{p_x^2}{2\gamma_0 m_0}\right]\right)$$

or $H(x,p_x) = \frac{\gamma_0 m_0 v_0^2 \kappa_0^2 x^2}{2} + \frac{p_x^2}{2\gamma_0 m_0} = \phi(x) + \frac{p_x^2}{2\gamma_0 m_0}$, $\phi(x) = -\int F_x dx$. This is identical to the non-relativistic Hamiltonian, with an effective mass of $\gamma_0 m_0$.

2. (a) Direct differentiation with respect to z of the square of the rms emittance most directly yields:

$$\frac{d}{dz}\varepsilon_{x,rms}^2 = \frac{d}{dz}\left[\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2\right]$$
$$= 2\langle xx'\rangle\langle x'^2\rangle + 2\langle x^2\rangle\langle x'x''\rangle - 2\langle xx'\rangle\langle x'^2\rangle - 2\langle xx'\rangle\langle xx''\rangle$$
$$= 2\langle x^2\rangle\langle x'x''\rangle - 2\langle xx'\rangle\langle xx''\rangle$$

Substitution of the functional form $F_x \propto x$, or in more standard form,

$x'' = \frac{F_x}{\gamma\beta^2 m_0 c^2} \equiv -\kappa_0^2 x$, we have

$$\frac{d}{dz}\varepsilon_{x,rms}^2 = -2\kappa_0^2\langle x^2\rangle\langle xx'\rangle - 2\kappa_0^2\langle xx'\rangle\langle x^2\rangle = 0.$$

(b) Direct differentiation of the square of the normalized emittance yields:

$$\frac{d}{dz}\varepsilon_{x,rms}^2 = \frac{1}{m_0^2c^2}\frac{d}{dz}\left[\left\langle x^2\right\rangle\left\langle p_x^2\right\rangle - \left\langle xp_x\right\rangle^2\right]$$

$$= \frac{2}{m_0^2c^2}\left[\frac{\left\langle xp_x\right\rangle}{p_z}\left\langle p_x^2\right\rangle + \left\langle x^2\right\rangle\left\langle p_xp_x'\right\rangle - \frac{\left\langle p_x^2\right\rangle}{p_z}\left\langle xp_x\right\rangle - \left\langle xp_x\right\rangle\left\langle xp_x'\right\rangle\right]$$

$$= \frac{2}{m_0^2c^2}\left[\left\langle x^2\right\rangle\left\langle p_xp_x'\right\rangle - \left\langle xp_x\right\rangle\left\langle xp_x'\right\rangle\right]$$

Again we may use the linear transverse force assumption, $p_x' = \frac{F_x}{\beta c} \equiv -\beta\gamma m_0 c\kappa_0^2 x$

and $\frac{d}{dz}\varepsilon_{x,rms}^2 = \frac{2\beta\gamma\kappa_0^2}{m_0c}\left[\left\langle x^2\right\rangle\left\langle xp_x\right\rangle - \left\langle xp_x\right\rangle\left\langle x^2\right\rangle\right] = 0$, *independent* of acceleration.

3. The change to the relativistic version of the Child-Langmuir law begins with the relation

$\frac{d^2\phi_e}{dz^2} = \frac{J_z}{\varepsilon_0 v_z}$, where J_z is constant in static flow. The relativisitic relation for the velocity is found by identifying the change in potential energy with the gain in kinetic, to give $v_z = c\sqrt{1-\left(\frac{e\phi_e}{m_0c^2}+1\right)^{-2}}$. We thus have $\frac{d^2\phi_e}{dz^2} = \frac{J_z}{\varepsilon_0 c\sqrt{1-\left(\frac{e\phi_e}{m_0c^2}+1\right)^{-2}}}$,

which can be integrated by first multiplying each side of the equation by $\frac{d\phi_e}{dz}$, to obtain:

$$\frac{d\phi_e}{dz} = \sqrt{\frac{J_z m_0 c}{\varepsilon_0}}\left[\left(\frac{e\phi_e}{m_0c^2}+1\right)^2 - 1\right]^{1/4} = \sqrt{\frac{J_z m_0 c}{\varepsilon_0}}\left[\left(\frac{e\phi_e}{m_0c^2}\right)^2 + 2\left(\frac{e\phi_e}{m_0c^2}\right)\right]^{1/4}.$$

The second integral of ϕ_e is more complicated, and involves use of a hypergeometric function. The derivation follows:

$$\frac{d\phi_e}{\left[\left(\frac{e\phi_e}{m_0c^2}\right)^2 + 2\left(\frac{e\phi_e}{m_0c^2}\right)\right]^{1/4}} = \sqrt{\frac{J_z m_0 c}{\varepsilon_0}}dz, \text{ or } \int\frac{d\phi_e}{\left[\left(\frac{e\phi_e}{m_0c^2}\right)^2 + 2\left(\frac{e\phi_e}{m_0c^2}\right)\right]^{1/4}} = \sqrt{\frac{J_z m_0 c}{\varepsilon_0}}d,$$

where *d* is the gap over which the voltage is applied. Solution for J_z gives the relativistic Child-Langmuir law.

4. The total amount of charge emitted comes from two contributions to the integral: the constant value of surface charge density inside the cutoff region, and the expected charge from the outer region.

The cutoff region extends from the origin to where $\sigma_{b0}\exp\left(-r_c^2/2\sigma_r^2\right)=\sigma_{bm}$, or $r_c=\sigma_r\sqrt{-2\ln\left(\sigma_{bm}/\sigma_{b0}\right)}$. The total charge emitted in this region is therefore $Q_1=\pi r_c^2\sigma_{bm}=2\pi\sigma_r^2\sigma_{bm}\ln\left(\sigma_{b0}/\sigma_{bm}\right)$, while the amount of charge emitted in the region outside of r_c is $Q_2=2\pi\sigma_{b0}\int_{r_c}^{\infty}\tilde{r}\exp\left(-\tilde{r}^2/2\sigma_r^2\right)d\tilde{r}=2\pi\sigma_{b0}\sigma_r^2\exp\left(-r_c^2/2\sigma_r^2\right)=2\pi\sigma_r^2\sigma_{bm}$.
The total charge emitted is $Q_{\text{emitted}}=Q_1+Q_2=2\pi\sigma_r^2\sigma_{bm}\left[1+\ln\left(\sigma_{b0}/\sigma_{bm}\right)\right]$, and normalizing to $Q_{\text{expected}}=2\pi\sigma_r^2\sigma_{b0}$, we have

$$\frac{Q_{\text{emitted}}}{Q_{\text{expected}}}=\frac{\sigma_{bm}}{\sigma_{b0}}\left[1+\ln\left(\frac{\sigma_{b0}}{\sigma_{bm}}\right)\right].$$

5. With the gradient of 20 MV/m, η=1, 100 A current, and 150 MeV, the invariant envelope is given by

$$\sigma_{inv}(\zeta,z)=\frac{1}{\gamma'}\sqrt{\frac{r_e\lambda}{(2+\eta)\gamma}}=\frac{m_0c^2}{eE_0}\sqrt{\frac{r_em_0c^2I}{3Uec}}$$

$$=\frac{0.511}{20}\sqrt{\frac{0.511\cdot 100}{3\cdot 150\cdot 1.7\times 10^4}}=66\ \mu\text{m}$$

The ratio of the space-charge term to the emittance term in the envelope equation $R=Ir_e\sigma_x^2/2\gamma\varepsilon_n^2ec=5.4\times 10^{-2}$. The beam is emittance dominated, and space-charge does not play a leading role in the rms spot size evolution.

6. (a) The optimum RF wavelength to choose to minimize the emittance at one nC can be found by differentiating the expression of the square of the emittance, with the charge factors conveniently set to unity:

$$\frac{\partial\varepsilon_n^2}{\partial\lambda_{\text{rf}}}=\tfrac{4}{3}a_1\left(\lambda_{\text{rf}}\right)^{1/3}+\tfrac{2}{3}a_2\left(\lambda_{\text{rf}}\right)^{-1/3}-\tfrac{2}{3}a_3\left(\lambda_{\text{rf}}\right)^{-5/3}=0.$$

Solution of this expression for our values a_i of occurs when $\lambda_{\text{rf}}\cong 9$ cm (very close to the S-band design of 10.5 cm), and the normalized emittance is $\varepsilon_n=0.69$ mm-mrad.
(b) Scaling of the brightness with charge is

$$B_e=\frac{2I}{\varepsilon_n^2}\propto\frac{Q(\text{nC})^{2/3}\lambda_{\text{rf}}(\text{m})^{-1}}{a_1\left(\frac{Q(\text{nC})}{\lambda_{\text{rf}}(\text{m})}\right)^{2/3}+a_2\left(\frac{Q(\text{nC})}{\lambda_{\text{rf}}(\text{m})}\right)^{4/3}+a_3\left(\frac{Q(\text{nC})}{\lambda_{\text{rf}}(\text{m})}\right)^{8/3}}.$$

One can see by inspection that this expression has no local minimum, but has an asymptotic minimum as the charge tends to zero, regardless of RF wavelength.

9. (a) For large numbers of cells, the relative mode separation (normalized to the RF frequency) may be estimated by

$$\kappa_c \left(\frac{\pi}{2N_c} \right)^2 = 0.05 \left(\frac{\pi}{24} \right)^2 = 8.6 \times 10^{-4}$$

This needs to be compared to the relative Q-width of the resonances, $\left(\frac{\Delta f}{f} \right) = \frac{1}{Q}$; we are demanding the mode separation with to be twice the Q-width, or $Q = \frac{2}{8.6 \times 10^{-4}} = 2300$. This width is typical of X-band RF structures.

(b) If we originally have 4 Q-width separation at a given RF frequency, when we scale up in RF frequency, $Q \propto \omega^{-1/2}$. We will be reduced to a 2 Q-width separation at twice the original RF frequency.

10. The bandwidth of the 266 nm laser light in a 30 fs pulse (assuming Fourier transform limit), is

$$\frac{\Delta\omega}{\omega} = \frac{1}{\omega \Delta t} = \frac{\lambda}{2\pi c \Delta t} = \frac{2.66 \cdot 10^{-7}\,\mathrm{m}}{2\pi \cdot 3 \cdot 10^{8}\,\mathrm{m/s} \cdot 3 \cdot 10^{-14}\,\mathrm{s}} = 4.7 \cdot 10^{-3}.$$

The spread in photon energy at this wavelength is $\Delta U = \hbar \Delta\omega = 0.022$ eV. This is a number typical of how much the nominal photon energy exceeds the work-function. Thus more (or less) energy is available for the photoelectrons. The quantum efficiency should thus rise; on the negative side, the “thermal” emittance of the photocathode should also rise.